The nonlinear eikonal relation of a weakly inhomogeneous magnetized plasma upon the action of arbitrarily polarized finite wavelength electromagnetic waves

V. Stefan[a] and N. A. Krall
JAYCOR, San Diego, California 92138-6580

J. B. McBride
Science Applications International Corporation, La Jolla, California 92038

(Received 1 April 1987; accepted 12 October 1987)

Based on a hydrodynamic Maxwell formalism of a weakly inhomogeneous magnetized plasma, a mode–mode coupling eikonal relation is derived. Finite wavelength and arbitrary polarization of a monochromatic driver pump have been taken into account as well as longitudinal and transverse components of the self-consistent plasma electric field. A weak turbulence theory method is used to solve the resulting inhomogeneous Volterra-type integral equation in tensorial form, i.e., an expansion of hydrodynamic quantities over the resultant electric field (driver pump and self-consistent field) up to the third order. The eikonal coupling relation thus obtained is discussed for the case of longitudinal and transverse interactions.

I. INTRODUCTION

In the last two decades strong attention has been focused on studies of dispersion features of plasma immersed in a strong electromagnetic wave (e.g., Refs. 1–3). The first derivations of the nonlinear mode–mode coupling dispersion relation were based on the interaction of an EM wave with plasma in the dipole approximation ($k_0 = 0$, where k_0 = wavenumber of the electromagnetic wave-driver pump). In this case it was possible to treat the initial system (hydrodynamical and Maxwell and/or kinetic and Maxwell) in the frames oscillating with the corresponding plasma components. This simplified the algebra and enabled one to obtain the dispersion relation in powers of E_0 (E_0 = intensity of the electric field of a driver pump). Taking into account the finite wavelength of a driver pump ($k_0 \neq 0$), the transformation into a pure oscillatory system is not possible because of the covariance of the wavelength with the chosen reference frame. In this paper the EM wave–magnetized plasma interaction will be treated in the laboratory reference frame utilizing the mathematical apparatus of weak turbulence theory. Magnetized plasma is treated in a hydrodynamic approximation allowing weak plasma inhomogeneity in the nonperturbed state. The self-consistent plasma field is described by a full set of Maxwell's equations allowing both longitudinal and transverse components in the plasma electric field.

In Sec. II, model equations for the interaction are presented. A hydrodynamic approximation with a general force term is adopted and the external electromagnetic field is assumed to be of arbitrary polarization. In Sec. III, a perturbation analysis of the hydrodynamic-Maxwell set of equations for a spatially and temporally dispersive plasma is presented. A definition of the nonlinear plasma dielectric permittivity tensor up to the nth order is given. The linear response of a weakly inhomogeneous plasma is addressed in Sec. III A. Since this work is based on a plasma dielectric permittivity formalism, the dielectric permittivity tensors for a weakly inhomogeneous plasma in both kinetic form and in the hydrodynamic approximation are given. They are subsequently reduced to corresponding forms describing longitudinal plasma eigenmodes. In Sec. III B, the nonlinear current density responses up to third order are evaluated. The plasma is assumed to be weakly inhomogeneous, so that plasma eigenmodes can be treated by a WKB approximation. The WKB treatment of plasma eigenmodes is given in Appendix B. The presentation is based on the concept of a mobility tensor for the "α" plasma component, which can be found in many textbooks (see for example, Ref. 4). A general nonlinear mode coupling relation is evaluated in Sec. IV, and discussed for both local and nonlocal effects. In Sec. V, longitudinal and transverse interactions are defined in a general way. Longitudinal nonlinear interactions are treated in Sec. V, and nonlinear transverse interactions in Sec. VI. In Sec. VII, applications of this powerful formalism are discussed.

II. MODEL EQUATIONS

The initial set of equations has the form

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n \mathbf{V}_\alpha) = 0, \tag{1}$$

$$\frac{\partial \mathbf{V}_\alpha}{\partial t} + \mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha = \frac{e_\alpha}{m_\alpha}\left(\mathbf{E} + \frac{\mathbf{V}_\alpha \times \mathbf{B}_0}{c}\right) + \frac{\mathbf{F}_\alpha}{m_\alpha}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

[a] Present address: S-Cubed, a division of Maxwell Laboratories, Inc., San Diego, California 92121.

 0031-9171/87/123703-10$01.90

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \mathbf{j} = \sum_\alpha e_\alpha n_\alpha \mathbf{V}_\alpha . \tag{6}$$

Equations (1) and (2) are the known hydrodynamic magnetized plasma equations with force term $\mathbf{F}_\alpha$. In linear approximation this term, which can be dependent on plasma inhomogeneity, temperature or temperature gradient, magnetic curvature, etc., drive MHD modes unstable.[5] Maxwell's equations are presented in terms of the electric ($\mathbf{D}$) and magnetic induction ($\mathbf{B}$) vectors. The system (1)–(6) is closed by $\mathbf{D} = f(\mathbf{E})$ or $\mathbf{D} = f(\mathbf{j})$ (see below). All quantities in the initial system are assumed to be functions of $\mathbf{r},t$ (position and time), and $\alpha \equiv e,i$ denotes electrons and ions, respectively.

Without the force term, Eqs. (1) and (2) are referred to as cold fluid equations. The role of F_α is as follows. In the zeroth approximation ($\mathbf{F}_\alpha^{(0)}$), it is connected to zeroth order velocity ($\mathbf{V}_\alpha^{(0)}$) defined here as drift velocity of the "α" plasma component resulting from the gradients in plasma density, temperature, and curvature of the external magnetic field (see below). We assume that these gradients are weak, so that terms containing $V_\alpha^{(0)}$ in higher order (second and third) approximations are neglected. Accordingly, the scope of the present paper is to give the nonlinear mode–mode coupling equation for plasma eigenmodes with weak gradients. In the zeroth approximation we can write (gradients are along the x axis and $\mathbf{B}_0$ along the z axis)

$$F_\alpha^{(0)} = -\nabla n_\alpha^{(0)} T_\alpha^{(0)}/n_\alpha^{(0)} - m_\alpha [(V_{T\alpha}^{(0)})^2/R_c^2]\mathbf{R}_c - m_\alpha \nu_{\alpha\beta}(\mathbf{V}_\alpha^{(0)} - \mathbf{V}_\beta^{(0)}). \tag{7a}$$

Here $n_\alpha^{(0)}$ and $T_\alpha^{(0)}$ are the plasma density and temperature (energy units) assumed to vary slowly in space and time. In Eq. (7a), $\mathbf{R}_c$ is the vectorized curvature radius of the external magnetic field and $\nu_{\alpha\beta}$ is the collision frequency between particles α and β. In the first approximation for the force term we have

$$F_\alpha^{(1)} \sim -\nabla p^{(1)}/n_\alpha^{(0)} - m_\alpha \nu_{\alpha\beta}(\mathbf{V}_\alpha^{(1)} - \mathbf{V}_\beta^{(1)}), \tag{7b}$$

where p is the plasma pressure. The plasma viscosity tensor is neglected in (1) and (2) so that the hydrodynamical system is closed by a properly chosen plasma state equation. Inclusion of the pressure gradient term in (7b) leads to the appearance of thermal effects (warm fluid) in the plasma dielectric permittivity. The Langevin (frictional) term gives dissipation in the dielectric permittivity resulting from collisions. Phenomenologically, plasma kinetic effects (Landau and cyclotron damping) can be included by formally writing the dielectric permittivity in kinetic form. A strict proof for this would consist of the replacement of the hydrodynamic equations in the system (1)–(6) by kinetic plasma equations and repetition of the calculations, beyond the scope of the present paper. In what follows, the plasma dielectric permittivity tensors will be considered as known, given within the framework of linear plasma electrodynamics. Accordingly, we give general expressions for dielectric permittivities in both hydrodynamic and kinetic form.

For the resultant electric and magnetic field we assume

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0(\mathbf{r},t) + \delta\mathbf{E}(\mathbf{r},t), \quad \delta\mathbf{E}(\mathbf{r},t) \ll \mathbf{E}_0(\mathbf{r},t), \\ \mathbf{B}(\mathbf{r},t) &= B_0(\mathbf{r}) + \delta\mathbf{B}(\mathbf{r},t), \quad \delta\mathbf{B}(\mathbf{r},t) \ll \mathbf{B}_0, \end{aligned} \tag{8}$$

where $\delta E(\mathbf{r},t)$ and $\delta B(\mathbf{r},t)$ describe the self-consistent plasma field and $E_0(\mathbf{r},t)$ denotes the driver pump. Here we shall assume arbitrary polarization of a driver pump so that $\mathbf{E}_0(\mathbf{r},t)$ has the following form:

$$\mathbf{E}_0(\mathbf{r},t) = \mathbf{E}_0 \cos(\omega_0 t - \mathbf{k}_0 \cdot \mathbf{r} + \delta_0) + \mathbf{E}_1 \cos(\omega_0 t - \mathbf{k}_0 \cdot \mathbf{r} + \delta_1). \tag{9}$$

The notation in (9) is conventional. In what follows we shall need the Fourier transform of (9), which is given by

$$\begin{aligned} \mathbf{E}_0(\omega,\mathbf{k}) &= \tfrac{1}{2}(\mathbf{E}_0 e^{i\delta_0} + \mathbf{E}_1 e^{i\delta_1})\delta(\omega - \omega_0)\delta(\mathbf{k} - \mathbf{k}_0) \\ &\quad + \tfrac{1}{2}(\mathbf{E}_0 e^{-i\delta_0} + \mathbf{E}_1 e^{-i\delta_1}) \\ &\quad \times \delta(\omega + \omega_0)\delta(\mathbf{k} + \mathbf{k}_0). \end{aligned} \tag{10}$$

From (9) we have three cases of a driver pump polarization.

(a) Linear polarization:

$$\delta_0 = \delta_1 = 0, \quad \mathbf{E}_1 \equiv 0. \tag{11}$$

(b) Circular polarization:

$$\mathbf{E}_0 \cdot \mathbf{E}_1 = 0, \quad |\mathbf{E}_0| = |\mathbf{E}_1|, \quad \delta_0 - \delta_1 = \pm \pi/2. \tag{12}$$

(c) Elliptic polarization:

$$\mathbf{E}_0 \cdot \mathbf{E}_1 = 0, \quad |\mathbf{E}_0| \neq |\mathbf{E}_1|, \quad \delta_0 - \delta_1 = \pm \pi/2. \tag{13}$$

In (12) and (13), + and − correspond to right and left helicity, respectively.

III. PERTURBATION ANALYSIS AND NONLINEAR PLASMA DIELECTRIC PERMITTIVITIES

Let us now introduce the constitutive relation $\mathbf{D} = f(\mathbf{E})$ or $\mathbf{D} = f(\mathbf{j})$, in the form

$$\mathbf{D}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) + 4\pi \int_{-\infty}^{t} \mathbf{j}(\mathbf{r},t')\,dt'. \tag{14}$$

Integration from $-\infty$ to "t" is a consequence of the causality principle. Knowing that Eqs. (1) and (2) give $\mathbf{j}(\mathbf{r},t) = f[\mathbf{E}(\mathbf{r},t)]$, (14) could be transformed, phenomenologically, into

$$D_i^L(\mathbf{r},t) = \int_{-\infty}^{t} dt_1 \int d\mathbf{r}_1 \times \varepsilon_{ij}(t - t_1, \mathbf{r} - \mathbf{r}_1, \mu_1 t, \mu_2 \mathbf{r}) E_j(\mathbf{r}_1, t_1). \tag{15}$$

In (15) we took into account that a collisionless magnetized plasma is anisotropic so that the dielectric permittivity (ε_{ij}) is a tensor. Dependence of ε_{ij} on $(t - t_1)$ and $(\mathbf{r} - \mathbf{r}_1)$ denotes temporal and spatial plasma dispersion, while dependence on $\mu_2 \mathbf{r}$ and $\mu_1 t$ describes a weakly inhomogeneous and quasistationary plasma. In what follows, however, plasma will be assumed to be stationary and weakly inhomogeneous so that the inhomogeneity scale length L_N is much larger than the scale length of the three-wave nonlinear interaction

$$L = \left(\frac{\partial \Delta k}{\partial x}\right)^{-1/2},$$

where $\Delta\mathbf{k} = \mathbf{k}_0 - \mathbf{k}_1 - \mathbf{k}_2$, and, in 1-D, $\Delta k = k_{0,x} - k_{1,x} - k_{2,x}$. Here $k_{\alpha,x}$, $\alpha = 0,1,2$, have algebraic values depending on the interaction geometry. In defining L it is assumed that the inhomogeneity gradient is along the "x" axis and $\mathbf{k}_1$ and $\mathbf{k}_2$ are wave vectors of plasma modes in interaction with the driver pump. We note that

$$\int d\mathbf{r} \equiv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\,dy\,dz$$

and that, obviously, (15) gives a linear dependence between $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$.

Utilizing the substitutions $t - t_1 \equiv t_1^*$ and $\mathbf{r} - \mathbf{r}_1 = \mathbf{r}_1^*$, and applying a Fourier transformation to (15), we obtain (the Einstein sum convention is used throughout)

$$D_i^L(\omega,\mathbf{k}) = \varepsilon_{ij}(\omega,\mathbf{k},\mathbf{r})E_j(\omega,\mathbf{k}), \tag{16}$$

where (weak inhomogeneous plasma)

$$\varepsilon_{ij}(\omega,\mathbf{k},\mathbf{r}) = \int_0^\infty dt_1^* \int d\mathbf{r}_1^*\, \varepsilon_{ij}(t_1^*,\mathbf{r}_1^*,\mathbf{r})e^{i\omega t_1^* - i\mathbf{k}\mathbf{r}_1^*}. \tag{17}$$

Equation (15) could be generalized to include nonlinearities with respect to $\mathbf{E}(\mathbf{r},t)$. Accordingly, we can write

$$D_i(\mathbf{r},t) = D_i^L(\mathbf{r},t) + D_i^{\mathrm{NL}}(\mathbf{r},t),$$

where

$$D_i^{\mathrm{NL}}(\mathbf{r},t) = \sum_{n=2}^\infty \int dt_1\cdots dt_n \int d\mathbf{r}_1\cdots d\mathbf{r}_n\, \varepsilon_{ij_1j_2,\dots,j_n} \times (t - t_1;t_1 - t_2;\dots;t_{n-1} - t_n;\mathbf{r} - \mathbf{r}_1; \mathbf{r}_1 - \mathbf{r}_2;\dots;\mathbf{r}_{n-1} - \mathbf{r}_n,\mathbf{r}) \prod_{l=1}^n E_{jl}(t_l,\mathbf{r}_l), \tag{18}$$

$$\int dt_n \equiv \int_{-\infty}^{t_{n-1}} dt_n.$$

Fourier transforming (18) gives

$$D_i^{\mathrm{NL}}(\omega,\mathbf{k},\mathbf{r}) = \sum_{n=2}^\infty \int d\omega_1\cdots d\omega_{n-1} \int d\mathbf{k}_1\cdots d\mathbf{k}_{n-1} \times \varepsilon_{ij_1j_2,\dots,j_n}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1;\dots;\omega_{n-1},\mathbf{k}_{n-1},\mathbf{r}) \times E_{j_1}(\omega - \omega_1,\mathbf{k} - \mathbf{k}_1)\cdots E_{j_{n-1}} \times (\omega_{n-2} - \omega_{n-1};\mathbf{k}_{n-2} - \mathbf{k}_{n-1})E_{j_n} \times (\omega_{n-1},\mathbf{k}_{n-1}). \tag{19}$$

From (19) a definition of nonlinear dielectric permittivity of nth index follows in the form

$$\varepsilon_{ij_1j_2,\dots,j_n}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1;\dots;\omega_{n-1},\mathbf{k}_{n-1}) = \int_0^\infty dt\, e^{i\omega t}\int_0^\infty dt_1\, e^{i\omega_1 t_1}\cdots\int_0^\infty dt_{n-1}\, e^{i\omega_{n-1}t_{n-1}} \times \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\int d\mathbf{r}_1\, e^{-i\mathbf{k}_1\cdot\mathbf{r}_1}\cdots\int d\mathbf{r}_{n-1}\, e^{-i\mathbf{k}_{n-1}\mathbf{r}_{n-1}} \times \varepsilon_{ij_1j_2,\dots,j_n}(t,\mathbf{r};t_1,\mathbf{r}_1;\dots;t_{n-1},\mathbf{r}_{n-1}). \tag{20}$$

For symmetry properties of the nonlinear dielectric permittivity given by (20), see Appendix A.

Let us now analyze Maxwell's equations. Applying the curl operator ∇ on (4) we obtain an equation describing the nonlinear propagation of plasma waves:

$$\nabla\times[\nabla\times\mathbf{E}(\mathbf{r},t)] + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}D^L(\mathbf{r},t) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}D^{\mathrm{NL}}(\mathbf{r},t). \tag{21}$$

The Fourier transform of (21) in tensorial form reads

$$\frac{c^2k^2}{\omega^2}\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)E_j(\omega,\mathbf{k}) - D_i^L(\omega,\mathbf{k},\mathbf{r}) = D_i^{\mathrm{NL}}(\omega,\mathbf{k}) \tag{22}$$

or

$$M_{ij}(\omega,\mathbf{k})E_j(\omega,\mathbf{k}) = -D_i^{\mathrm{NL}}(\omega,\mathbf{k}), \tag{23}$$

where M_{ij} is the Maxwell plasma tensor

$$M_{ij} = \varepsilon_{ij}(\omega,\mathbf{k}) - \frac{c^2k^2}{\omega^2}\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right). \tag{24}$$

Then, a solution of the tensor equation (23) is

$$E_j(\omega,\mathbf{k}) = E_j^{(0)}(\omega,\mathbf{k}) - A_{ja}D_a^{\mathrm{NL}}(\omega,\mathbf{k}). \tag{25}$$

In (25) A_{ja} is the inverse Maxwell tensor defined by $A_{ja}(\omega,\mathbf{k})M_{as}(\omega,\mathbf{k}) = \delta_{js}$ (δ_{js} = Kronecker delta) and $E_j^{(0)}(\omega,\mathbf{k})$ is the solution of linear electrodynamics ($D_i^{\mathrm{NL}} \equiv 0$).

Using the perturbation method[6,7] (weak turbulence approach) in (1) and (2), that is expanding hydrodynamic quantities over the electric field

$$\begin{aligned} n(\mathbf{r},t) &= n_0(\mu_1t,\mu_2\mathbf{r}) + n^{(1)}(\mathbf{r},t) + n^{(2)}(\mathbf{r},t) + n^{(3)}(\mathbf{r},t),\dots,\\ \mathbf{V}(\mathbf{r},t) &= \mathbf{V}_0(\mu_1t,\mu_2\mathbf{r}) + V^{(1)}(\mathbf{r},t) + V^{(2)}(\mathbf{r},t) + \mathbf{V}^{(3)}(\mathbf{r},t),\dots,\\ \mathbf{j}(\mathbf{r},\mathbf{k}) &= \mathbf{j}_0(\mu_1t,\mu_2\mathbf{r}) + \mathbf{j}^{(1)}(\mathbf{r},t) + \mathbf{j}^{(2)}(\mathbf{r},t) + \mathbf{j}^{(3)}(\mathbf{r},t),\dots, \end{aligned} \tag{26}$$

we obtain a system of iterative equations describing the nonlinear processes up to the third order with respect to the resultant electric field (driver pump field and self-consistent field). The nonperturbed state (zeroth approximation) is defined by n_0, $\mathbf{V}_0$, and $\mathbf{j}_0$, and according to (26) is weakly disturbed by the external and self-consistent plasma fields.The smallness indicators μ_1 and μ_2 are meant to emphasize that the slow variation of the equilibrium in space (as a result of inhomogeneity) and time (resulting from mode coupling, instability, etc.) are, in general, not small to the same order. In the present calculation, we retain only first order variation in $u_2\mathbf{r}$.

A. Linear current response

In the zeroth approximation we have

$$\frac{\partial n_0}{\partial x} \neq 0, \quad L_N = \left(\frac{1}{n_0}\frac{\partial n_0}{\partial x}\right)^{-1}, \tag{27}$$

and for the unperturbed hydrodynamic velocity $\mathbf{V}_0$, $\mathbf{V}_0 \equiv \mathbf{V}_D$, where $\mathbf{V}_D$ is the drift velocity vector as a result of the force term $\mathbf{F}_\alpha$. The drift velocity resulting from gradients in unperturbed density ($n_\alpha^{(0)}$) and temperature ($T_\alpha^{(0)}$, energy units) is[8]

$$\mathbf{V}_D^{(\alpha)} = (c/e_\alpha n_\alpha^{(0)}B_0^2)[\mathbf{B}_0\times\nabla(n_\alpha^{(0)}T_\alpha^{(0)})],$$

and in the case of magnetic field curvature[4,5]

$$\mathbf{V}_D^{(\alpha)} = (cm_\alpha/e_\alpha B_0^2)(\mathbf{B}_0\times\nabla\ln B_0)(V_{\alpha\perp}^2/2 + V_{\alpha\parallel}^2).$$

In the above expressions it was assumed that the magnetic field is along the z axis and gradients of $n_\alpha^{(0)}, T_\alpha^{(0)}$ along the x axis. In the first order approximation from (1) and (2) we obtain

$$\frac{\partial n_\alpha^{(1)}}{\partial t} + n_\alpha^{(0)}\nabla V_\alpha^{(1)} + n_\alpha^{(1)}\nabla V_\alpha^{(0)} + V_\alpha^{(1)}\nabla n_\alpha^{(0)} = 0, \qquad (28)$$

$$\frac{\partial V_\alpha^{(1)}}{\partial t} - \frac{e_\alpha}{m_\alpha c}(V_\alpha^{(1)}\times\mathbf{B}_0) - (\mathbf{V}_0\nabla)\mathbf{V}_\alpha^{(1)} = \frac{e_\alpha}{m_\alpha}\mathbf{E}(\mathbf{r},t). \qquad (29)$$

System (28) and (29) with Maxwell equations (3)–(6) describe eigenoscillations ($E_0\equiv 0$) of weakly inhomogeneous plasma in the hydrodynamic appoximation (see, e.g., Ref. 5 and references therein). Using (28) and (29), we can obtain eikonal equations for plasma eigenmodes (see Appendix B):

$$|k^2\delta_{ij} - k_i k_j - (\omega^2/c^2)\varepsilon_{ij}(\omega,\mathbf{k},\mathbf{r})| = 0. \qquad (29')$$

In (29′), $\varepsilon_{ij}(\omega,\mathbf{k},\mathbf{r})$ is the usual dielectric permittivity tensor of a weakly inhomogeneous magnetized plasma in hydrodynamic appoximation. From (29′) we can obtain $\mathbf{k}=\mathbf{k}(\mathbf{r})$ (for given ω) and accordingly the eikonal function $\psi(\mathbf{r}) = \int \mathbf{k}(\mathbf{r})\cdot d\mathbf{r}$. Note that in the case of homogeneous plasma, $\varepsilon(\omega,\mathbf{k},\mathbf{r})\rightarrow\varepsilon(\omega,\mathbf{k})$ and (29′) is referred to as a dispersion relation, giving $\omega=\omega(\mathbf{k})$, the frequency spectrum for given $\mathbf{k}$. From (28) and (29) we have

$$n_\alpha^{(1)} = i(e_\alpha/m_\alpha)n_0(k_i/\omega)\Gamma_{ij}^{(\alpha)}(\omega,\mathbf{r})E_j(\omega,\mathbf{k}), \qquad (30)$$

$$V_{\alpha,i}^{(1)} = i(e_\alpha/m_\alpha)\Gamma_{ij}^{(\alpha)}(\omega,\mathbf{r})E_j(\omega,\mathbf{k}), \qquad (31)$$

where

$$\Gamma_{ij}^{(\alpha)}(\omega,\mathbf{r}) = -(\omega^2/\omega_{p\alpha}^2)\chi_{ij}^{(\alpha)}(\omega,\mathbf{r}). \qquad (32)$$

Here $\chi_{ij}^{(\alpha)}(\omega,\mathbf{r})$ is the linear susceptibility of the "α" plasma component including weak inhomogeneity. It is given in general form as[8] (the inhomogeneity gradient assumed to be along the x axis)

$$\chi_{ij}^{(\alpha)} = \sum_\alpha \frac{1}{T_\alpha}\left(1 - \frac{k_y V_{T\alpha}^2}{\omega\Omega_\alpha}\frac{\partial}{\partial x}\right) T_\alpha\,[\epsilon_{ij}^{(\alpha)}(\omega,\mathbf{k},x) - \delta_{ij}],$$

where $\epsilon_{ij}^{(\alpha)}$ is the partial tensor of the dielectric permittivity of the "α" plasma component. It can be found in standard textbooks on plasma physics (see, for example, Refs. 4 and 8) allowing plasma parameters to depend on x. In the case of longitudinal eigenmodes of a weakly inhomogeneous Maxwellian plasma, the kinetic form of ϵ_{ij} is

$$\epsilon_{ij}^{(\alpha)}(\omega,\mathbf{k},x) \rightarrow \epsilon^{(\alpha)}(\omega,\mathbf{k},x) = 1 + [\omega_{p\alpha}^2(x)/k^2V_{T\alpha}^2(x)] \times\left(1 - \sum_{n=-\infty}^{\infty}\frac{\omega}{\omega - n\Omega_\alpha(x)} \times A_n[z_\alpha(x)]T_+[\beta_{n\alpha}(x)]\right),$$

where

$$T_+(x) = x\exp\left(-\frac{x^2}{2}\right)\int_{i\infty}^{x}\exp\left(\frac{t^2}{2}\right)dt,$$

$$z_\alpha(x) = [k_\perp\rho_\alpha(x)]^2,\quad \beta_{n\alpha}(x) = \frac{\omega - n\Omega_\alpha(x)}{|k_z|V_{T\alpha}(x)}.$$

Here, Ω_α and $V_{T\alpha}$ are the cyclotron frequency and thermal speed of the "α" particle, respectively, and $\rho_\alpha(x)$ is the "α" particle Larmor radius. The hydrodynamic approximation, which would follow from Eqs. (1)–(6), follows from $z_\alpha\rightarrow 0$, $\beta_{n\alpha}\rightarrow\infty$, $\omega\rightarrow\omega+i\nu_\alpha$. The power of using the permittivity is that either the kinetic or hydrodynamic form can be used without altering the formalism.

Note that $T_+(x)$ can be expressed in terms of tabulated Kramp functions $W(x)$ in the form

$$T_+^{(\alpha)}[\beta_{n\alpha}(x)] = -i\sqrt{\pi/2}\beta_{n\alpha}(x)W(\beta_{n\alpha}(x)/\sqrt{2}).$$

Also, $A_n^{(\alpha)}(z_\alpha) = \exp(-z_\alpha)I_n(z_\alpha)$, where I_n is a modified Bessel function of the order "n." The solution of Eq. (29) could be written, also, in the form

$$V_{\alpha,i}^{(1)}(\omega,\mathbf{k}) = \mu_{ij}^{(\alpha)}(\omega,\mathbf{k};x)E_j(\omega,\mathbf{k}), \qquad (33)$$

which gives the definition of the mobility tensor of the "α" plasma component $\mu_{ij}^{(\alpha)}(\omega,\mathbf{k},x)$. From (32) and (33) we have

$$\mu_{ij}^{(\alpha)}(\omega,\mathbf{k};x) = -i(e_\alpha/m_\alpha)(\omega/\omega_{p\alpha}^2)\chi_{ij}^{(\alpha)}(\omega,\mathbf{k};x). \qquad (34)$$

The current density in linear approximation is

$$j^{(1)}(\mathbf{r},t) = \sum_\alpha e_\alpha n_\alpha^{(1)}V_\alpha^{(0)} + \sum_\alpha e_\alpha n_\alpha^{(0)}V_\alpha^{(1)}. \qquad (35)$$

After applying the Fourier transformation (FT) we obtain

$$j_\alpha^{(1)}(\omega,\mathbf{k}) = \zeta_{ij}^{(\alpha)}(\omega,\mathbf{k};x)E_j(\omega,\mathbf{k}), \qquad (36)$$

where $\zeta_{ij}^{(\alpha)}(\omega,\mathbf{k};x)$ is the conductivity tensor of the "α" plasma component given by

$$\zeta_{ij}^{(\alpha)}(\omega,\mathbf{k};x) = -i(e_\alpha^2 n_0/m_\alpha)(\omega/\omega_{p\alpha}^2)\chi_{ij}^{(\alpha)}(\omega,\mathbf{k};x). \qquad (37)$$

B. Nonlinear current response

In the second order approximation we have

$$\frac{\partial V_\alpha^{(2)}}{\partial t} - \frac{e_\alpha}{m_\alpha c}\mathbf{V}_\alpha^{(2)}\times\mathbf{B}_0 + (\mathbf{V}_\alpha^{(0)}\nabla)\mathbf{V}_\alpha^{(2)} + (V_\alpha^{(2)}\nabla)V_\alpha^{(0)} = \frac{e_\alpha}{m_\alpha}\mathbf{V}_\alpha^{(1)}\times\mathbf{B}^{(1)} - (\mathbf{V}_\alpha^{(1)}\nabla)\mathbf{V}_\alpha^{(1)}, \qquad (38)$$

$$\frac{\partial n_\alpha^{(2)}}{\partial t} + n_\alpha^{(0)}\nabla\mathbf{V}_\alpha^{(2)} + \mathbf{V}_\alpha^{(2)}\nabla n_\alpha^{(0)} + \mathbf{V}_\alpha^{(1)}\nabla n_\alpha^{(1)} + n_\alpha^{(1)}\nabla\mathbf{V}_\alpha^{(1)} = 0. \qquad (39)$$

In (38) and (39) we shall neglect terms in $V_\alpha^{(0)}$ as being third order terms, since $V_\alpha^{(0)}\equiv V_D = V_\alpha^{(0)}(\nabla n_\alpha,\nabla T_\alpha,\nabla B_0)$ and gradients are assumed weak. We note that the left-hand side of (38) is analogous to the left-hand side of (29), and consequently, (38) could be represented in the form [see (33)]

$$V_{\alpha,i}^{(2)}(\omega,\mathbf{k})\cdot\theta_{ij}^{(\alpha)}(\omega,\mathbf{k},\mathbf{r}) = A_j(\omega,\mathbf{k}), \qquad (40)$$

where $A_j(\omega,\mathbf{k})$ is the FT of the jth component of the vector on the right-hand side of (38) which has the form

$$\mathrm{FT}[(\mathbf{V}^{(1)}\nabla)\mathbf{V}^{(1)}]_\alpha = i\int d\omega_1\,d\mathbf{k}_1[V_s^{(1)}(\omega_1,\mathbf{k}_1)(k_s - k_{1,s})]V_\alpha^{(1)}(\omega_2,\mathbf{k}_2), \qquad (41)$$

$$\mathrm{FT}\left(\frac{e_\alpha}{m_\alpha c}\mathbf{V}_\alpha^{(1)}\times\mathbf{B}^{(1)}\right) = \frac{e_\alpha}{m_\alpha c}\int d\omega_1\,d\mathbf{k}_1\,\mathbf{V}_\alpha^{(1)}(\omega_1,\mathbf{k}_1)\times\mathbf{B}^{(1)}(\omega_2,\mathbf{k}_2), \qquad (42)$$

where

$$\mathbf{B}^{(1)}(\omega_2,\mathbf{k}_2) = (c/\omega_2)\mathbf{k}_2\times\mathbf{E}^{(1)}(\omega_2,\mathbf{k}_2). \quad (43)$$

Also,

$$\theta_{ij}^{-1} = (m_\alpha/e_\alpha)\mu_{ij} = i\Gamma_{ij}.$$

Taking this into account, after some algebra we obtain

$$V_{\alpha,i}^{(2)}(\omega,\mathbf{k}) = -\frac{e_\alpha}{m_\alpha}\int d\omega_1\, d\mathbf{k}_1\, d\omega_2\, d\mathbf{k}_2\, \delta(\omega-\omega_1-\omega_2)\delta \times(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\frac{E_j(\omega_2,\mathbf{k}_2)E_s(\omega_1,\mathbf{k}_1)}{\omega_1\omega_2} \times[-\omega\Gamma_{ij}(\omega)\Gamma_{as}(\omega_1)k_{2,a} + \omega_1\Gamma_{js}(\omega_1)\Gamma_{ia} \times(\omega)k_{2,a} + \omega_2\Gamma_{ij}(\omega_2)\Gamma_{as}(\omega_1)k_{2,a}]. \quad (44)$$

For the nonlinear current density $j^{(2)}(\mathbf{r},t)$ we have

$$\mathbf{j}^{(2)}(\mathbf{r},t) = \sum_\alpha e_\alpha n_\alpha^{(1)}\mathbf{V}_\alpha^{(1)} + \sum_\alpha e_\alpha n_0 \mathbf{V}_\alpha^{(2)}, \quad (45)$$

and taking into account (30), (31), and (44), we finally obtain

$$j_{\alpha,i}^{(2)} = -\frac{e_\alpha^3 n_0}{m_\alpha}\int d\omega_1\, d\mathbf{k}_1\, d\omega_2\, d\mathbf{k}_2\, \delta(\omega-\omega_1-\omega_2)\delta \times(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\frac{E_j(\omega_2,\mathbf{k}_2)E_s(\omega_1,\mathbf{k}_1)}{\omega_1\omega_2} \times[\omega_2\Gamma_{ij}(\omega_2)\Gamma_{as}(\omega_1)k_{2,a} + \omega_1\Gamma_{js}(\omega_1)\Gamma_{ia}(\omega)k_{2,a} - \omega\Gamma_{ij}(\omega)\Gamma_{as}(\omega_1)k_{2,a}]. \quad (46)$$

We note that

$$j_\alpha^{(2)}(\omega,\mathbf{k}) = -i\frac{\omega}{4\pi}\int d\omega_1\, d\omega_2\, d\mathbf{k}_1\, d\mathbf{k}_2\, \delta(\omega-\omega_1-\omega_2)\delta \times(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\varepsilon_{ijs}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1) \times E_j(\omega_2,\mathbf{k}_2)E_s(\omega_1,\mathbf{k}_1),$$
$$\omega_2 = \omega-\omega_1, \quad \mathbf{k}_2 = \mathbf{k}-\mathbf{k}_1. \quad (47)$$

Taking into account the symmetry property $\varepsilon_{ijs}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1) = \varepsilon_{isj}(\omega,\mathbf{k};\omega_2,\mathbf{k}_2)$ [see (A6)] we can introduce a new tensor S_{ijs} (Pustovalov–Silin formalism, see Ref. 6) defined as

$$S_{ijs}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1) \equiv \varepsilon_{ijs}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1) + \varepsilon_{isj}(\omega,\mathbf{k};\omega_2,\mathbf{k}_2). \quad (48)$$

From (48), it follows that $S_{ijs} \equiv 2\varepsilon_{i(js)}$, where $\varepsilon_{i(js)}$ is the symmetric tensor of tensor ε_{ijs}.[9]

In addition to $\mathbf{V}_\alpha^{(2)}$ we have to know the nonlinear density $n_\alpha^{(2)}(\omega,\mathbf{k})$, which could be easily found from the continuity equation

$$n_\alpha^{(2)}(\omega,\mathbf{k}) = \mathbf{k}\cdot\mathbf{j}_\alpha^{(2)}(\omega,\mathbf{k})/e_\alpha m. \quad (49)$$

To third order the hydrodynamic equation (2) reads

$$\frac{\partial\mathbf{V}_\alpha^{(3)}}{\partial t} - \frac{e_\alpha}{m_\alpha c}\mathbf{V}_\alpha^{(3)}\times\mathbf{B}_0 + (\mathbf{V}_0\nabla)\mathbf{V}^{(3)} + (V_\alpha^{(3)}\nabla)V_\alpha^{(0)} = -(\mathbf{V}_\alpha^{(1)}\nabla)\mathbf{V}_\alpha^{(2)} - (\mathbf{V}_\alpha^{(2)}\nabla)\mathbf{V}_\alpha^{(1)} + \frac{e_\alpha}{m_\alpha c}\mathbf{V}_\alpha^{(2)}\times\mathbf{B}^{(1)} - (V_\alpha^{(2)}\nabla)V_\alpha^{(0)}. \quad (50)$$

Here again we neglect the terms with $V_\alpha^{(0)}$ for the reasons mentioned above. Using a similar method as in the case of Eq. (38), we obtain

$$\mathbf{V}_{\alpha,i}^{(3)} = -\frac{e_\alpha^3}{m_\alpha^3}\Gamma_{ia}(\omega)\int d\omega_1\, d\mathbf{k}_1\, d\omega_2\, d\mathbf{k}_2\, \frac{E_j(\omega-\omega_1,\mathbf{k}-\mathbf{k}_1)E_s(\omega_1-\omega_2,\mathbf{k}_1-\mathbf{k}_2)E_r(\omega_2,\mathbf{k}_2)}{\omega_2(\omega_1-\omega_2)}$$
$$\times\left(\frac{k_a-k_{1,a}}{\omega-\omega_1}\delta_{jc} - \delta_{aj}\frac{k_c-k_{1,c}}{\omega-\omega_1} + k_{1,b}\Gamma_{bj}(\omega-\omega_1)\delta_{ac} + (k_c-k_{1,c})\Gamma_{aj}(\omega-\omega_1)\right)$$
$$\times\{(\omega_1-\omega_2)\Gamma_{rs}(\omega_1-\omega_2)\Gamma_{cd}(\omega_1)k_{2,d} + \Gamma_{ds}(\omega_1-\omega_2)k_{2,d}[\omega_2\Gamma_{cr}(\omega_2) - \omega_1\Gamma_{cr}(\omega_1)]\}. \quad (51)$$

For the nonlinear current $j_\alpha^{(3)}(\mathbf{r},t)$ we have

$$j_\alpha^{(3)}(\mathbf{r},t) = e_\alpha n_0 V_\alpha^{(3)}(\mathbf{r},t) + e_\alpha n^{(1)}(\mathbf{r},t)V_\alpha^{(2)}(\mathbf{r},t) + e_\alpha n_\alpha^{(2)}(\mathbf{r},t)V_\alpha^{(1)}(\mathbf{r},t). \quad (52)$$

Using (51), (44), (31), (30), and (49) after FT we obtain

$$j_{\alpha,i}^{(3)}(\omega,k) = -i\frac{e_\alpha^4 n_0}{m_\alpha^3}\int d\omega_1\, d\mathbf{k}_1\, d\omega_2\, d\mathbf{k}_2\, \frac{E_j(\omega-\omega_1,\mathbf{k}-\mathbf{k}_1)E_s(\omega_1-\omega_2,\mathbf{k}_1-\mathbf{k}_2)E_r(\omega_2,\mathbf{k}_2)}{\omega_1\omega_2(\omega-\omega_1)(\omega_1-\omega_2)}$$
$$\times\{(\omega-\omega_1)\Gamma_{ij}(\omega-\omega_1)k_{1,b}\Gamma_{br}(\omega_2)\omega_2(k_{1,a}-k_{2,a})\Gamma_{as}(\omega_1-\omega_2) + [\delta_{jb}(k_a-k_{1,a})\omega_1\Gamma_{ia}(\omega)$$
$$+\delta_{ib}(k_a-k_{1,a})\omega_1\Gamma_{aj}(\omega-\omega_1) + (\omega-\omega_1)k_b\Gamma_{ij}(\omega-\omega_1) - \omega(k_b-k_{1,b})\Gamma_{ij}(\omega)$$
$$+(\omega-\omega_1)k_{1,a}\Gamma_{aj}(\omega-\omega_1)\omega_1\Gamma_{ib}(\omega)][(\omega_1-\omega_2)\Gamma_{rs}(\omega_1-\omega_2)\Gamma_{bc}(\omega_1)k_{2,c}$$
$$+\Gamma_{cs}(\omega_1-\omega_2)k_{2,c}\omega_2\Gamma_{br}(\omega_2) - \Gamma_{cs}(\omega_1-\omega_2)k_{2,c}\omega_1\Gamma_{br}(\omega_1)]\}. \quad (53)$$

It is to be noted that (53) implicitly gives the nonlinear dielectric permittivity to third order $\varepsilon_{ijsr}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2,\mu_2 x)$.

IV. NONLINEAR MODE–MODE COUPLING EIKONAL EQUATION

Up to this point we have given a quite general derivation which represents the basic methodology in nonlinear plasma physics. The nonlinear current (53) is usually used for studying the nonlinear interaction of plasma waves up to third order, where E_j, E_s, E_r are the electric fields of plasma waves.[6,7] For our purpose, however, we shall assume that E_j, E_s, E_r are the resultant field consisting of the external component E_0 (driver pump) and the self-consistent component δE (plasma field). In a triple product of resultant electric field, we shall retain only the terms quadratic to E_0 and linear to δE. This will lead to the relation (see the following) describing nonlinear (E_0^2) coupling of linear (δE) plasma eigenperturbations. Consequently, taking into account Eq. (9) we have for the triple product,

$$E_s(\omega_1,\mathbf{k}_1)E_j(\omega_2,\mathbf{k}_2)E_r(\omega_3,\mathbf{k}_3) \sim \tfrac{1}{4}(\mathbf{E}_0e^{i\delta_0} + \mathbf{E}_1e^{i\delta_1})_c(\mathbf{E}_0e^{-i\delta_0} + \mathbf{E}_1e^{-i\delta_1})_a\delta E_b(\omega,\mathbf{k})$$
$$\times\{\delta_{bj}[\delta_{cs}\delta_{ar}\delta(\omega_1-\omega_0)\delta(\mathbf{k}_1-\mathbf{k}_0)\delta(\omega_3+\omega_0)\delta(\mathbf{k}_3+\mathbf{k}_0) + \delta_{cr}\delta_{as}\delta(\omega_1+\omega_0)\delta(\mathbf{k}_1+\mathbf{k}_0)\delta(\omega_3-\omega_0)\delta(\mathbf{k}_3-\mathbf{k}_0)]$$
$$+\delta_{br}[\delta_{cs}\delta_{aj}\delta(\omega_1-\omega_0)\delta(\mathbf{k}_1-\mathbf{k}_0)\delta(\omega_2+\omega_0)\delta(\mathbf{k}_2+\mathbf{k}_0) + \delta_{cj}\delta_{as}\delta(\omega_1+\omega_0)\delta(\mathbf{k}_1+\mathbf{k}_0)\delta(\omega_2-\omega_0)\delta(\mathbf{k}_2-\mathbf{k}_0)]$$
$$+\delta_{bs}[\delta_{cj}\delta_{ar}\delta(\omega_2-\omega_0)\delta(\mathbf{k}_2-\mathbf{k}_0)\delta(\omega_3+\omega_0)\delta(\mathbf{k}_3+\mathbf{k}_0) + \delta_{cr}\delta_{aj}\delta(\omega_2+\omega_0)\delta(\mathbf{k}_2+\mathbf{k}_0)\delta(\omega_3-\omega_0)\delta(\mathbf{k}_3-\mathbf{k}_0)]\}. \qquad (54)$$

Utilizing (54) and (53) from (22) we obtain (see also Ref. 10)

$$\varepsilon_{ij}(\omega,\mathbf{k}) - (c^2k^2/\omega^2)(\delta_{ij} - k_ik_j/k^2)\delta E_j(\omega,\mathbf{k})$$
$$= -\tfrac{1}{2}S_{ijs}(\omega,\mathbf{k};\omega-\omega_0,\mathbf{k}-\mathbf{k}_0)(\mathbf{E}_0e^{i\delta_0} + \mathbf{E}_1e^{-i\delta_1})_j$$
$$\times\delta E_s(\omega-\omega_0,\mathbf{k}-\mathbf{k}_0) - \tfrac{1}{2}S_{ijs}(\omega,\mathbf{k};\omega+\omega_0,\mathbf{k}+\mathbf{k}_0)$$
$$\times(\mathbf{E}_0e^{i\delta_0} + \mathbf{E}_1e^{-i\delta_1})_j\delta E_s(\omega+\omega_0,\mathbf{k}+\mathbf{k}_0). \qquad (55)$$

Here S_{ijs} is given by (48) and in explicit form reads

$$S_{ijs}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1)$$
$$= -i(e_\alpha/m_\alpha)(\omega_{p\alpha}^2/\omega\omega_1\omega_2)\{\omega_1k_n\Gamma_{nj}(\omega_2)\Gamma_{is}(\omega_1)$$
$$+\omega_2k_n\Gamma_{ns}(\omega_1)\Gamma_{ij}(\omega_2) + \omega_1k_{2,n}\Gamma_{in}(\omega)\Gamma_{js}(\omega_1)$$
$$+\omega_2k_{1,n}\Gamma_{in}(\omega)\Gamma_{sj}(\omega_2) - \omega k_{2,n}\Gamma_{ns}(\omega_1)\Gamma_{ij}(\omega)$$
$$-\omega k_{1,n}\Gamma_{nj}(\omega_2)\Gamma_{is}(\omega)\}. \qquad (56)$$

Eliminating $\delta E_i(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0)$ from (55) [see Eq. (25)]

$$\delta E_i(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0)$$
$$= -\tfrac{1}{2}A_{ia}(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0)S_{ajs}(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0;\omega,\mathbf{k})$$
$$\times(E_0e^{\pm i\delta_0} + E_1e^{\pm i\delta_1})_j\delta E_s(\omega,\mathbf{k}), \qquad (57)$$

where

$$A_{ia}(\omega,\mathbf{k}) = [\varepsilon_{ij}(\omega,\mathbf{k}) - c^2k^2/\omega^2(\delta_{ij} - k_ik_j/k^2)]^{-1}, \qquad (58)$$

we obtain nonlinear dielectric permittivity (nonlinear with respect to external field E_0^2, and linear with respect to plasma perturbations, δE)

$$\tilde{\varepsilon}_{ij}(\omega,k) = \varepsilon_{ij}(\omega,k) - \tfrac{1}{4}[S_{ias}(\omega,\mathbf{k};\omega+\omega_0,\mathbf{k}+\mathbf{k}_0)A_{sc}$$
$$\times(\omega+\omega_0,\mathbf{k}+\mathbf{k}_0)S_{cbj}(\omega+\omega_0,\mathbf{k}+\mathbf{k}_0;\omega,\mathbf{k})$$
$$+S_{ibs}(\omega,\mathbf{k};\omega-\omega_0,\mathbf{k}-\mathbf{k}_0)A_{sc}(\omega-\omega_0,\mathbf{k}-\mathbf{k}_0)$$
$$\times S_{caj}(\omega-\omega_0,\mathbf{k}-\mathbf{k}_0;\omega,\mathbf{k})]$$
$$\times(\mathbf{E}_0e^{i\delta_0} + \mathbf{E}_1e^{-i\delta_1})_a(\mathbf{E}_0e^{i\delta_0} + \mathbf{E}_1e^{-i\delta_1})_b. \qquad (59)$$

Assuming that $\omega_0 \gg \omega$, from (59) we have

$$S_{ijs}(\omega,\mathbf{k};\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0) \sim -\sum_\alpha \frac{ie_\alpha}{m_\alpha\omega_0}\delta_{sj}k_m\chi_{im}(\omega,\mathbf{k}), \qquad (60)$$

and for the inverse Maxwell plasma tensor

$$A_{sc}(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0)$$
$$= \frac{1}{\Delta^\pm}\left\{\frac{(\mathbf{k}\pm\mathbf{k}_0)_s(\mathbf{k}\pm\mathbf{k}_0)_c}{(\mathbf{k}\pm\mathbf{k}_0)^2\varepsilon(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0)} + \left[\delta_{cs} - \frac{(\mathbf{k}\pm\mathbf{k}_0)_s(\mathbf{k}\pm\mathbf{k}_0)_c}{(\mathbf{k}\pm\mathbf{k}_0)^2}\right]\left[\varepsilon(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0) - \frac{c^2(\mathbf{k}\pm\mathbf{k}_0)^2}{(\omega\pm\omega_0)^2}\right]^{-1}\right\}, \qquad (61)$$

where

$$\Delta^\pm = 1 + \frac{k^2}{4}\chi_e(\omega,\mathbf{k})\left(\frac{[(\mathbf{k}\pm\mathbf{k}_0)\mathbf{r}_{EB}]^2}{(\mathbf{k}\pm\mathbf{k}_0)^3\varepsilon(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0)} + \frac{[(\mathbf{k}\pm\mathbf{k}_0)\times\mathbf{r}_{EB}]^2}{(\mathbf{k}\pm\mathbf{k}_0)^2[\varepsilon(\omega\pm\omega_0,\mathbf{k}\pm\mathbf{k}_0) - c^2(\mathbf{k}\pm\mathbf{k}_0)^2/(\omega\pm\omega_0)^2]}\right). \qquad (62)$$

Substituting (60) and (61) into (59) we obtain the dielectric permittivity for low-frequency plasma mode with longitudinal and transverse electric field. If we are interested only in the longitudinal low-frequency $(\omega,\mathbf{k})$ plasma mode, the longitudinal contraction of (59) has to be evaluated as

$$(k_ik_j/k^2)\tilde{\varepsilon}_{ij}(\omega,\mathbf{k},\mathbf{r}),$$

which gives the final nonlinear mode–mode coupling eikonal equation

$$\tilde{\varepsilon}(\omega,\mathbf{k},\mathbf{r}) = \frac{\varepsilon_L(\omega,\mathbf{k},\mathbf{r})}{\chi_e(\omega,\mathbf{k},\mathbf{r})[1+\chi_i(\omega,\mathbf{k},r)]} + \frac{k^2(|(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}_{EB}|)^2}{4(\mathbf{k}-\mathbf{k}_0)^3\varepsilon_L(\omega-\omega_0,\mathbf{k}-\mathbf{k}_0,\mathbf{r})}$$
$$+\frac{k^2(|(\mathbf{k}+\mathbf{k}_0)\cdot\mathbf{r}_{EB}|)^2}{4(\mathbf{k}+\mathbf{k}_0)^2\varepsilon_L(\omega-\omega_0,\mathbf{k}-\mathbf{k}_0,\mathbf{r})} + \frac{k^2(\omega-\omega_0)^2(|(\mathbf{k}-\mathbf{k}_0)\times\mathbf{r}_{EB}|)^2}{4(\mathbf{k}-\mathbf{k}_0)^2[(\omega-\omega_0)^2\varepsilon_T(\omega-\omega_0,\mathbf{k}-\mathbf{k}_0,\mathbf{r})-c^2(\mathbf{k}-\mathbf{k}_0)^2]}$$
$$+\frac{k^2(\omega+\omega_0)^2(|(\mathbf{k}+\mathbf{k}_0)\times\mathbf{r}_{EB}|)^2}{4(\mathbf{k}+\mathbf{k}_0)^2[(\omega+\omega_0)^2\varepsilon_T(\omega+\omega_0,\mathbf{k}+\mathbf{k}_0,\mathbf{r})-c^2(\mathbf{k}+\mathbf{k}_0)^2]} = 0. \quad (63)$$

Here $\mathbf{r}_{EB}$ is the vector amplitude of electron oscillations with respect to ions in the magnetic field B_0 and the driver pump electric field (9). Consequently,

$$\mathbf{r}_{EB} = \mathbf{r}_{EB}(\mathbf{E}_0 e^{i\delta_0}) + \mathbf{r}_{EB}(\mathbf{E}_1 e^{i\delta_1}). \quad (64)$$

To treat arbitrary polarization we can choose $\delta_0 \equiv 0$, $\delta_1 = \pm\pi/2$ so that

$$\mathbf{r}_{EB} = \mathbf{r}_{EB}(\mathbf{E}_0) \pm i\mathbf{r}_{EB}(\mathbf{E}_1), \quad (65)$$

where

$$\mathbf{r}_{EB}(\mathbf{E}) = \left|\begin{matrix} E_x \dfrac{e}{m_e}\dfrac{1}{\omega_0^2-\Omega_e^2} - \dfrac{e_i}{m_i}\dfrac{1}{\omega_0^2-\Omega_i^2} - iE_y\dfrac{e}{m_e}\dfrac{\Omega_e}{\omega_0(\omega_0^2-\Omega_e^2)} - \dfrac{e_i}{m_i}\dfrac{\Omega_i}{\omega_0(\omega_0^2-\Omega_i^2)} \\ iE_x\dfrac{e}{m_e}\dfrac{\Omega_e}{\omega_0(\omega_0^2-\Omega_e^2)} - \dfrac{e_i}{m_i}\dfrac{\Omega_i}{\omega_0(\omega_0^2-\Omega_i^2)} + E_y\dfrac{e}{m_e}\dfrac{1}{\omega_0^2-\Omega_e^2} - \dfrac{e_i}{m_i}\dfrac{1}{\omega_0^2-\Omega_i^2} \\ E_z\dfrac{e}{m_e\omega_0^2} - \dfrac{e_e}{m_i\omega_0^2} \end{matrix}\right|,$$

with $\mathbf{E}\equiv\mathbf{E}_0;\mathbf{E}_1$.

Inserting (59) into (29b), we obtain an eikonal mode–mode coupling equation allowing the low-frequency mode $(\omega,\mathbf{k})$ to have both longitudinal and transverse components of the electric field. This is also valid for high-frequency modes ($\omega\pm\omega_0$; sidebands). If the low-frequency mode is purely longitudinal, the eikonal equation is given by (63). According to (63), the low-frequency longitudinal mode is coupled to mixed (longitudinal and transverse) high-frequency modes (sidebands). Frequency and damping (growth) rate of oscillations are found by solving (63) with respect to k_x. Then

$$\int_{x_1}^{x_2} \mathrm{Re}\, k_{x,s}(\omega,\mathbf{E}_0,\mathbf{k}_0,x)dx = \pi\left(n+\frac{1}{2}\right), \quad (67)$$

$$\gamma = \int_{x_1}^{x_2} \mathrm{Im}\, k_{x,s}(\omega,\mathbf{E}_0,\mathbf{k}_0,x)dx \times\left(\int_{x_1}^{x_2}\frac{\partial}{\partial\omega}\mathrm{Re}\, k_{x,s}(\omega,E_0,\mathbf{k}_0,x)dx\right)^{-1}. \quad (68)$$

Integration is over the transparent region between turning points defined by

$$\mathrm{Re}\, k_{x,s}^2(\omega,\mathbf{E}_0,\mathbf{k}_0,x) = 0, \quad (69)$$

where s denotes a particular branch of oscillations. Relation (67) is usually called the quasiclassical quantization rule or Bohr–Sommerfeld phase integral.

From another point of view, (63) could be considered as a local dispersion relation giving the local frequency spectrum and the local growth (damping) rate in the case of very weak plasma inhomogeneity ($kL_N, k_0L_N \gg 1$) and relatively strong damped modes. This is equivalent to considering $\varepsilon^{0,\pm,1} = 0$ and $D^{\pm} = 0$ as local dispersion relations in (63). In the opposite case, for relatively long wavelength and weakly damped modes, the region of transparency is comparable to the inhomogeneity scale length, and nonlocal effects [(67) and (68)] are essential.

V. STOKES AND ANTI-STOKES LONGITUDINAL SIDEBAND COUPLING

Let us now interpret relation (63). First, decomposing the electric field into a longitudinal and transverse component

$$E_i = (k_ik_j/k^2)E_j + (\delta_{ij} - k_ik_j/k^2)E_j, \quad (70)$$

and inserting it into (22) (with $D^{\mathrm{NL}}\equiv 0$), we have

$$[(c^2k^2/\omega^2)(\delta_{ij}-k_ik_j/k^2)-\varepsilon_{ij}]E_j^T - \varepsilon_{ij}^L E_j^L = 0. \quad (71)$$

From (71) we see that $[(c^2k^2/\omega^2)(\delta_{ij}-k_ik_j/k^2)-\varepsilon_{ij}]$ characterizes the electromagnetic properties of plasma with respect to the transverse field and ε_{ij}^L with respect to the longitudinal field. Consequently, in (63) the second and third terms describe the coupling of low-frequency longitudinal $[\varepsilon_L(\omega,\mathbf{k})=0]$ modes to Stokes' and anti-Stokes' longitudinal modes. The third and fourth terms describe the scattering of Stokes' and anti-Stokes' transverse modes on the low-frequency longitudinal plasma mode. Accordingly, the longitudinal resonant interaction in (61) is defined by

$$\varepsilon_L^{\pm,0} = 0 \quad (72)$$

and the four wave coupling equation for resonant modes $[\varepsilon(\omega,\mathbf{k},\mathbf{r})\sim 0]$ is given by

$$\varepsilon_L(\omega,\mathbf{k},x) + \frac{\chi_e(\omega,\mathbf{k},x)[1+\chi_i(\omega,\mathbf{k},x)]k^2}{4(\mathbf{k}-\mathbf{k}_0)^2} \times \left(\frac{(|\mathbf{k}_-\cdot\mathbf{r}_{EB}|)^2}{\varepsilon(\omega_-,k_-,x)} + \frac{(|\mathbf{k}_+\cdot\mathbf{r}_{EB}|)^2}{\varepsilon(\omega_+,k_+,x)}\right) = 0, \quad (73)$$

where the wave four-vector $k_\pm \equiv (\mathbf{k}_\pm,\omega_\pm)$ is given by $k_\pm = (\mathbf{k}\pm\mathbf{k}_0,\omega\pm\omega_0)$. If only the Stokes sideband is resonant $[\varepsilon(\omega_-,k_-,x)\sim 0]$, Eq. (73) describes decay process. If both sidebands are resonant, Eq. (73) describes oscillating two stream instabilities $(k>k_0)$ and modulation instabilities $(k<k_0)$, as four wave coupling processes.

VI. FOUR WAVE RAMAN AND BRILLOUIN STIMULATED SCATTERING

Transverse resonant interaction is defined by

$$D^\pm \equiv \varepsilon^\pm - c^2(\mathbf{k}\pm\mathbf{k}_0)^2/(\omega\pm\omega_0)^2 = 0, \quad D^\pm \equiv D^T, \quad (74)$$

and the corresponding four wave coupling equation is given by

$$\varepsilon_L(\omega,\mathbf{k},\mathbf{r}) + \frac{\chi_e(\omega,\mathbf{k},x)[1+\chi_i(\omega,\mathbf{k},x)]k^2}{4} \cdot \left(\frac{\omega_-^2(|\mathbf{k}_-\times\mathbf{r}_{EB}|)^2}{[\omega_-^2\varepsilon(\omega_-,\mathbf{k}_-,x_-)-c^2k_-^2]k_-^2} + \frac{\omega_+^2(|\mathbf{k}_+\times\mathbf{r}_{EB}|)^2}{[\omega_+^2\varepsilon(\omega_+,\mathbf{k}_+,x_-)-c^2k_+^2]k_+^2}\right) = 0. \quad (75)$$

If $\varepsilon_L(\omega,\mathbf{k},x)$ corresponds to high-frequency inhomogeneous plasma modes, Eq. (75) describes four wave Raman stimulated scattering. If, however, $\varepsilon_L(\omega,\mathbf{k},x)$ corresponds to low-frequency inhomogeneous plasma modes, Eq. (75) describes four wave Brillouin stimulated scattering. In Eq. (75), $\varepsilon_{+,-}$ is the transverse plasma dielectric permittivity defined by the first term in Eq. (71), describing high-frequency plasma modes. Furthermore, if only Stokes' $(D_-\sim 0)$ or anti-Stokes' $(D_+\sim 0)$ components are resonant in a particular system, we have Raman and Brillouin Stokes' or anti-Stokes' (upconversion) scattering, respectively, as the three wave transverse interaction. In the case $\mathbf{k}\|\mathbf{k}_0$, the four wave coupling equation (75) can be used for studying the longitudinal modulation of an incident electromagnetic wave in a weakly inhomogeneous plasma. The case $\mathbf{k}\perp\mathbf{k}_0$ corresponds to transverse modulation of a driver pump (self-focusing and/or filamentation).

VII. CONCLUSION

In conclusion, we have given a general mathematical formalism for the treatment of nonlinear interactions of arbitrarily polarized, finite wavelength electromagnetic fields with weakly inhomogeneous plasma. The assumption of weak inhomogeneity enables a WKB treatment of plasma eigenmodes (Refs. 4 and 5). An inhomogeneous Volterra-type integral equation in tensorial form [Eq. (22)] is treated using a weak turbulence approximation. Using symmetry properties of the n-index nonlinear dielectric permittivity (Appendix A), a relatively simple mode–mode coupling relation is obtained. We note that this type of mode–mode coupling relation was obtained in Ref. 11 using a ponderomotive force formalism for homogeneous isotropic plasma with linearly polarized electromagnetic waves. Homogeneous magnetized plasma was addressed in Ref. 6, but only for the case $\omega_0 \gg \Omega_e$, also using weak turbulence theory.

The possibilities of using Eq. (63) in the treatment of the interaction of electromagnetic waves with weakly inhomogeneous plasma are enormous. It has been used for the EBT environment in Ref. 12 and in the tokamak environment in Ref. 13. Stabilization of interchange modes in mirrors using Eq. (63) was studied in Refs. 14–16. In addition, computational treatment of Eq. (63) would allow treatment of even more realistic interactions.

Related work to that discussed in this manuscript has recently been independently carried out elsewhere.[17] That work uses heuristic means to obtain some of the general expressions which are rigorously derived here. However, Ref. 17 examines applications of the formalism to compare with previous derivations, and demonstrates agreement in all cases, thus making a contribution which is complementary to that of the present paper.

ACKNOWLEDGMENTS

One of us (V.S.) would like to thank Dr. A. R. Wilson and Dr. D. E. Parks for their continuing interest in this research.

We would also like to thank Dr. B. Cohen for his preprint UCRL-96056, submitted to The Physics of Fluids.

This work was supported by the United States Department of Energy.

APPENDIX A: SYMMETRY PROPERTIES OF THE NONLINEAR DIELECTRIC PERMITTIVITY $\varepsilon_{ij_1,j_2,\ldots,j_n}$

In linear theory for a nondispersive and isotropic medium, the connection between the electric induction (**D**) vector and the electric field (**E**) is given in the form, $\mathbf{D}(\mathbf{r},t) = \varepsilon(\mathbf{r},t)\mathbf{E}(\mathbf{r},t)$. Here $\varepsilon(\mathbf{r},t)$ is the dielectric permittivity of an inhomogeneous and nonstationary medium. If the medium is anisotropic (external magnetic field), the relation is written in tensorial form, $D_i(\mathbf{r},t) = \varepsilon_{ij}(\mathbf{r},t)E_j(\mathbf{r},t)$. In both cases, however, there is a so-called local dependence between **D** and **E** (local constitutive relation). In the case of a dispersive medium, processes in different physical points, $(\mathbf{r},t)$, are dependent, and the connection between $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$ is nonlocal. A generalization of the above linear dependences gives

$$D_i(\mathbf{r},t) = \int_{-\infty}^{t} dt_1 \int d\mathbf{r}_1\, \varepsilon_{ij}(t_1,\mathbf{r}_1;t,\mathbf{r})E_j(\mathbf{r}_1,t_1). \quad (A1)$$

If the medium is weakly inhomogeneous and nonstationary, (A1) reduces to (15). In analogy to (A1) and (15), in order to take into account nonlinearities, we can write expression (18). In (18) for the case of quadratic nonlinearities $(n=2)$ we have $\varepsilon_{ij_1j_2}$ describing the influence of the electric processes in point $(\mathbf{r}_2,t_2)$ on processes in $(\mathbf{r}_1,t_1)$, and processes in $(\mathbf{r}_1,t_1)$ on processes in $(\mathbf{r},t)$. However, $\varepsilon_{ij_2j_1}$ describes a succession of processes in the following form: $(\mathbf{r}_1,t_1)\to(\mathbf{r}_2,t_2)\to(\mathbf{r},t)$. A similar discussion could be performed for cubic $(n=3)$ and higher nonlinearities of plasma. Now, naturally, appears the question of the relation between $\varepsilon_{ij_1j_2}$ and $\varepsilon_{ij_2j_1}$, etc. The symmetry properties of the nonlinear dielectric permittivity could be evaluated from (18). For the case of $\varepsilon_{ij_1j_2}$ we have

$$D_i(\mathbf{r},t) = \int dt_1 \int d\mathbf{r}_1\, \varepsilon_{ij_1j_2} \times (\mathbf{r}-\mathbf{r}_1,t-t_1,\mathbf{r}_1-\mathbf{r}_2,t_1-t_2) \times E_{j_1}(\mathbf{r}_1,t_1)E_{j_2}(\mathbf{r}_2,t_2). \quad \text{(A2)}$$

After Fourier transformation we obtain

$$D_i(\omega,\mathbf{k}) = \int d\omega_1 \int d\mathbf{k}_1\, \varepsilon_{ij_1j_2}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1) \times E_{j_1}(\omega-\omega_1,\mathbf{k}-\mathbf{k}_1)E_{j_2}(\omega_2,\mathbf{k}_2). \quad \text{(A3)}$$

Similarly, for $\varepsilon_{ij_2j_1}$ we have

$$D_i(\mathbf{r},t) = \int dt_1 \int d\mathbf{r}_1\, \varepsilon_{ij_2j_1} \times (\mathbf{r}-\mathbf{r}_2,t-t_2;\mathbf{r}_2-\mathbf{r}_1,t_2-t_1) \times E_{j_2}(\mathbf{r}_2,t_2)E_{j_1}(\mathbf{r}_1,t_1), \quad \text{(A4)}$$

and after FT

$$D_i(\omega,\mathbf{k}) = \int d\omega_1 \int d\mathbf{k}_1\, \varepsilon_{ij_2j_1}(\omega_1,\mathbf{k};\omega-\omega_1,\mathbf{k}-\mathbf{k}_1) \times E_{j_1}(\omega-\omega_1,\mathbf{k}-\mathbf{k}_1)E_{j_2}(\omega_1,\mathbf{k}_1). \quad \text{(A5)}$$

From (A3) and (A5) it follows that

$$\varepsilon_{ij_1j_2}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1) = \varepsilon_{ij_2j_1}(\omega,\mathbf{k};\omega-\omega_1,\mathbf{k}-\mathbf{k}_1). \quad \text{(A6)}$$

For the three-index nonlinear dielectric permittivity we have

$$\begin{aligned}&\varepsilon_{ij_1j_2j_3}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)\\ &= \varepsilon_{ij_3j_2j_1}(\omega,\mathbf{k};\omega-\omega_2,\mathbf{k}-\mathbf{k}_2;\omega-\omega_1,\mathbf{k}-\mathbf{k}_1)\\ &= \varepsilon_{ij_1j_3j_2}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1;\omega_1-\omega_2,\mathbf{k}_1-\mathbf{k}_2)\\ &= \varepsilon_{ij_2j_1j_3}(\omega,\mathbf{k};\omega+\omega_2-\omega_1,\mathbf{k}+\mathbf{k}_2-\mathbf{k}_1;\omega_2,\mathbf{k}_2). \quad \text{(A7)}\end{aligned}$$

Finally, as in the case of linear dielectric permittivity,

$$\begin{aligned}&\varepsilon^*_{ij_1j_2\ldots j_n}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2,\ldots,\omega_{n-1},\mathbf{k}_{n-1})\\ &= \varepsilon_{ij_1j_2\ldots j_n}(-\omega,-\mathbf{k};-\omega_1,-\mathbf{k}_1;\ldots;\\ &\quad -\omega_{n-1},-\mathbf{k}_{n-1}) \quad \text{(A8)}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Re}\,\varepsilon_{ij_1j_2\ldots j_n}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1;\ldots;\omega_{n-1},\mathbf{k}_{n-1})\\ &= \mathrm{Re}\,\varepsilon_{ij_1j_2\ldots j_n}(-\omega,-\mathbf{k};-\omega_1,-\mathbf{k}_1;\ldots;\\ &\quad -\omega_{n-1},-\mathbf{k}_{n-1}), \quad \text{(A9)}\end{aligned}$$

$$\begin{aligned}&\mathrm{Im}\,\varepsilon_{ij_1j_2\ldots j_n}(\omega,\mathbf{k};\omega_1,\mathbf{k}_1;\ldots;\omega_{n-1},\mathbf{k}_{n-1})\\ &= -\mathrm{Im}\,\varepsilon_{ij_1j_2\ldots j_n}(-\omega,-\mathbf{k};-\omega_1,-\mathbf{k}_1;\ldots;\\ &\quad -\omega_{n-1},-\mathbf{k}_{n-1}). \quad \text{(A10)}\end{aligned}$$

The asterisk denotes complex conjugate, and Re and Im the real and imaginary parts, respectively. Symmetry properties of the conductivity tensor in relativistic plasma have been recently addressed in Ref. 18.

APPENDIX B: A GENERAL EIKONAL TREATMENT OF THE NONLINEAR PROPAGATION EQUATIONS OF PLASMA WAVES

Here we use the general eikonal method of Weinberg[19] and apply it to the nonlinear propagation equation of plasma waves (21) in a somewhat different manner, namely, using explicitly the language of a multiple space scale perturbation technique. We start with the integrodifferential equation (21) in $(\omega,\mathbf{k})$ space given by (23). Formally, Eq. (23), excluding the inhomogeneous (nonlinear D^{NL}) term, is identical to Eq. (107) of Ref. 19. The slow space variation of the Maxwell plasma tensor $M_{ij}(\omega,\mathbf{k};\mu\mathbf{r})$ suggests that all quantities in Eq. (23), in addition to the fast space-time variables $(\mathbf{r},t)$, have a slow space variable $(\mathbf{r},t;\mu\mathbf{r})$. Accordingly, we formally write $E(\mathbf{r},t)$, $D^L(r,t)$, $D^{\mathrm{NL}}(\mathbf{r},t) \to a(r,t)$, where

$$a(\mathbf{r},t) \to a(\mathbf{r},t;\mu\mathbf{r},\mu^2\mathbf{r},\ldots) \equiv a(\mathbf{r},t;\mathbf{r}_1,\mathbf{r}_2,\ldots).$$

We seek the formal solution of Eq. (23) in the eikonal approximation

$$a(\mathbf{r},t;\mu,\mathbf{r}) = \sum_{e=0}^{\infty} \mu^e A_e(\mu\mathbf{r})e^{-i\omega t + i\psi(\mathbf{r},\mu\mathbf{r})}, \quad \text{(B1)}$$

where the eikonal function $\psi(\mathbf{r},\mu\mathbf{r})$ is defined by

$$\nabla_{\mathbf{r}}\psi(\mathbf{r},\mu\mathbf{r}) = \mathbf{k}(\mu\mathbf{r}). \quad \text{(B2)}$$

In (B2) $\nabla_{\mathbf{r}}$ denotes the gradient operator with respect to the fast variable $\mathbf{r}$. Now Eq. (23) can be rewritten in the form

$$M_{ij}(\omega,-i\nabla_{\mathbf{r}}-i\mu\nabla_{\mathbf{r}_1}-\cdots,\mu\mathbf{r})E_j(\omega,\mu\mathbf{r}) = -\varepsilon D_i^{\mathrm{NL}}(\omega,\mu\mathbf{r}). \quad \text{(B3)}$$

In (B3), ε denotes the smallness of the nonlinear term. In the zeroth approximation with respect to μ from (B3) we have

$$M_{ij}(\omega,-i\nabla_{\mathbf{r}})E_{j0}(\omega,\mu\mathbf{r}) = 0. \quad \text{(B4)}$$

Considering (B4) as an eigenvalue problem we have

$$M_{ij}(\omega,-i\nabla_{\mathbf{r}})E_{j0}(\omega,\mu\mathbf{r}) = M_{ij}(\omega,\mathbf{k}(\mu\mathbf{r}))E_{j0}(\omega,\mu\mathbf{r}), \quad \text{(B5)}$$

where the eigenvalues (for inhomogeneous plasma local eigenoscillations) are given by $\det M_{ij}(\omega,\mathbf{k}(\mu\mathbf{r}))=0$, which is actually the expression (29'). In the 1-D case, as assumed in this paper, this equation uniquely determines $\mathbf{k}(\mu\mathbf{r})$. The equation for the eigenfunctions (amplitudes, see also Ref. 20) is obtained as the first approximation with respect to μ in the form (WKB approximation)[19]

$$\begin{aligned}&\mu M_{ij}(\mu\mathbf{r},\omega,-i\nabla_{\mathbf{r}})E_{j1}(\mu\mathbf{r})\exp[-i\omega t + i\psi(\mathbf{r},\mu\mathbf{r})]\\ &\quad -\mu i\nabla_{\mathbf{k}}M_{ij}(\omega,\mathbf{k},\mu\mathbf{r})\nabla_{\mathbf{r}_1}E_{j0}(\mu\mathbf{r})\exp[-i\omega t + i\psi(\mathbf{r},\mu\mathbf{r})]\\ &\quad = -\varepsilon D_i^{\mathrm{NL}}(\omega,\mu\mathbf{r}). \quad \text{(B6)}\end{aligned}$$

In (B6) the derivative of M with respect to $\mu\mathbf{r}$ is neglected. Although formally the second term on the left-hand side of (B6) is of the same order as the nonlinear term, in our formalism it is neglected. This is a result of the fact that the nonlinear coupling in our case is induced by a relatively strong driver pump $(\varepsilon > \mu)$ and we assumed weak spatial dispersion. Accordingly, the space behavior of the amplitude is completely governed by the nonlinear term rather than by the slow spatial variation of plasma. In the opposite case both terms govern spatial amplitude behavior.

[1]V. P. Silin, *Parametric Action of High Power Radiation on a Plasma* (in Russian) (Nauka, Moscow, 1973).
[2]P. K. Kaw, W. L. Kruer, C. S. Liu, and K. Nishikawa, in *Advances in Plasma Physics*, edited by A. Simon and W. B. Thomson (Interscience, New York, 1976), Vol. 6, pp. 3–26.
[3]V. Stefan and N. A. Krall, Phys. Fluids **28**, 2937 (1985).
[4]A. B. Mikhailovskii, *Theory of Plasma Instabilities* (Consultants Bureau, New York, 1974), Vol. 2, pp. 24–28.
[5]N. A. Krall, in *Advances in Plasma Physics* (Interscience, New York, 1968), Vol. 1, p. 153.
[6]V. V. Pustovalov and V. P. Silin, *Teoriya Plazmi, Trudi Fian* (Nauka, Moscow, 1972), Vol. 61, p. 42 [also available in English: *Theory of Plasmas, Proceedings of the Lebedev Physics Institute* (Consultants Bureau, New York, 1975), Vol. 61, p. 37].
[7]B. B. Kadomtsev, *Plasma Turbulence* (Academic, New York, 1965).
[8]A. F. Alexandrov, L. S. Bogdankievich, and A. A. Rukhadze, *Principles of Plasma Electrodynamics* (in Russian) (Vishaya Shkola, Moscow, 1978), p. 243.
[9]G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1970), pp. 121–135.
[10]L. M. Gorbunov, Zh. Eksp. Teor. Fiz. **55**, 2298 (1968).
[11]J. F. Drake, P. K. Kaw, Y. C. Lee, G. Schmidt, C. S. Lie, and M. Rosenbluth, Phys. Fluids **17**, 778 (1974).
[12]V. Stefan and N. A. Krall, Phys. Rev. Lett. **52**, 41 (1984).
[13]V. Stefan and A. Bers, Phys. Fluids **27**, 175 (1984).
[14]J. B. McBride, V. Stefan, and N. A. Krall, Phys. Rev. Lett. **54**, 42 (1985).
[15]V. Stefan and J. B. McBride, Phys. Fluids **28**, 3189 (1985).
[16]J. B. McBride and V. Stefan, Phys. Fluids **29**, 1181 (1986).
[17]B. I. Cohen, Phys. Fluids **30**, 2676 (1987).
[18]H. E. Brandt, J. Math. Phys. **24**, 1332 (1983).
[19]S. Weinberg, Phys. Rev. **126**, 1899 (1962).
[20]I. B. Bernstein, Phys. Fluids **18**, 320 (1975); I. B. Bernstein and L. Friedland, *Basic Plasma Physics I*, edited by A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1983), pp. 368–449.